Conformal Field Theory and Gravity

Solutions to Problem Set 5 Fall 2024

1. Conserved quantitites and spinning string

(a) The Polyakov action in conformal gauge $g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}$ takes the form

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X_\mu \partial^\alpha X^\mu \tag{1}$$

Given a symmetry of the lagrangian $\mathcal{L}$ acting infinitesimally on the fields as $X^\mu \to X^\mu + \delta X^\mu$, Noether's theorem ensures we have conserved charges

$$j^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)}\delta X^\mu \tag{2}$$

In the case of Poincaré symmetry we have conserved currents for any choice of $\Lambda^\mu{}_\nu, c^\mu$, thus yielding

$$P^\alpha_\mu = T\partial^\alpha X_\mu \quad \text{and} \quad J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu \tag{3}$$

(b) The conserved momenta and angular momenta are found by integrating the currents on the spatial slices of the worldsheet.

$$P^\mu = \int d\sigma P^\mu_\tau = T\int d\sigma \partial_\tau X^\mu \tag{4}$$

$$J^{\mu\nu} = T\int d\sigma \; \partial_\tau X^\mu X^\nu - \partial_\tau X^\nu X^\mu \tag{5}$$

Plugging the mode expansion for the closed string

$$X^\mu = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}[\alpha^\mu_n e^{-in(\tau-\sigma)} + \tilde{\alpha}^\mu_n e^{-in(\tau+\sigma)}] \tag{6}$$

yields

$$P^\mu = \; p^\mu, \quad J^{\mu\nu} = l^{\mu\nu} + L^{\mu\nu} \tag{7}$$

where

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{8}$$

$$L^{\mu\nu} = \; -i\sum_{n>0}\frac{1}{n}[\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n] + (\alpha_n \leftrightarrow \tilde{\alpha}_n) \tag{9}$$

It is clear then that $l^{\mu\nu}$ and $L^{\mu\nu}$ represent the extrinsic and intrinsic angular momenta of the string respectively.

For the open string it works analogously, but now $L^{\mu\nu}$ only contains one set of oscillators. You should use the mode expansion

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}[\alpha_n^\mu e^{-in(\tau-\sigma)} + \alpha_n^\mu e^{-in(\tau+\sigma)}] \tag{10}$$

(Note the factor of 2 in front of p^μ).

(c) The equations of motion and constraints are trivially verified in position space at early times, although one could easily find the solution for all times by employing the expansion 6.
However, that is not necessary since the energy can be obtained from the knowledge of X^0 alone.

$$E = \int_0^{2\pi} d\sigma\ T\partial_\tau X^0 = 2\pi RT \tag{11}$$

Hence T is the ratio between the string energy and its length, which justifies its interpretation as the string tension

(d) The energy is given by

$$E = \int_0^{\pi} d\sigma\ T\partial_\tau X^0 = \pi RT \tag{12}$$

while the only independent component of the angular momentum is

$$J^{12} = T\int_0^\pi d\sigma\ \partial_\tau X^1 X^2 - \partial_\tau X^2 X^1 = TR^2 \int_0^\pi \cos^2\sigma(\sin^2\tau + \cos^2\tau) = \pi R^2 T/2 \tag{13}$$

Therefore, $J = J^{12}$ and thus $\frac{J}{E^2} = \frac{1}{2\pi T} = \alpha'$.
This is the maximum ratio for any string configuration since the string does not have any translational kinetic energy, and it is spinning at the maximum rate, given that the endpoint move at the speed of light:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\Big|_{\sigma=0,\pi} = 1 \tag{14}$$

2. T-duality and D-branes

(a) We impose $\alpha_0^\mu = \tilde{\alpha}_0^\mu$ because of the periodic boundary conditions

$$X^\mu(\sigma = 2\pi) - X^\mu(\sigma = 0) = \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)2\pi = 0 \tag{15}$$

(b) Since X^μ and $X^\mu + 2\pi nR$, $n \in \mathbb{Z}$ represent the same space-time point, $X^\mu(\sigma = 2\pi) - X^\mu(\sigma = 0)$ has to vanish up to $2\pi mR$,

$$X^\mu(\sigma = 2\pi) - X^\mu(\sigma = 0) = \sqrt{\frac{\alpha'}{2}}(\alpha_0^{25} - \tilde{\alpha}_0^{25})2\pi = 2\pi mR \tag{16}$$

However, this indicates that X^{25} has been winding around the 25th direction m times.

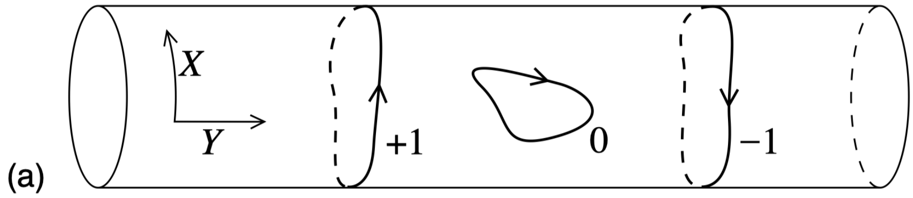

Figure 1: Closed strings with winding numbers $m = 1, 0, -1$. Here, X denotes the compactified dimension and Y the transverse noncompact dimensions.

(c) The condition on p^{25} is

$$2\pi R p^{25} = 2\pi n \implies \frac{n}{R} = p^{25} = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu) \tag{17}$$

Combining this with the previous condition, we obtain

$$\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} + m\frac{R}{\alpha'}\right) \tag{18}$$

$$\tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} - m\frac{R}{\alpha'}\right) \tag{19}$$

(d) The quantum mechanical interpretation of this compactification is the following. Instead of having continuous momentum in the 25'th direction, the string states have now labels n and m, and they can be excited by α_{-n}^μ, $\tilde{\alpha}_{-n}^\mu$ for $\mu = 0, ..., 24$. We still need to impose $L_0 - 1 = 0$ and $\tilde{L}_0 - 1 = 0$ on physical states, where $1 = a$ is the ordering ambiguity constant and L_0, $\tilde{L}_0$ are taken to be normal-ordered (i.e. annihilations operators are put on the right of the expressions). Defining the 25-dimensional mass $M^2 = \sum_{\mu=0}^{24} p_\mu p^\mu$. The above conditions can be rearranged to

$$N_R - N_L = nm \tag{20}$$

$$\alpha' M^2 = \alpha'\left(\left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2\right) + 2N_L + 2N_R - 4 \tag{21}$$

where N_L and N_R are the occupation numbers of the α's and $\tilde{\alpha}$'s respectively. We observe that M^2 is invariant under T-duality.

(e) The split is obvious after writing $\tau = \frac{1}{2}(\sigma^+ + \sigma^-)$, $\sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$ in the original mode expansion. We can write

$$X = X_L(\sigma + \tau) + X_R(\sigma - \tau) \tag{22}$$

where

$$X_L = x^{25} + \tilde{x}^{25} + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\tilde{\alpha}_m^\mu}{m}\bar{e}^{-im\sigma^+} \tag{23}$$

$$X_R = x^{25} - \tilde{x}^{25} + \sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\tilde{\alpha}_m^\mu}{m}\bar{e}^{-im\sigma^-} \tag{24}$$

Under T-duality, $\tilde{\alpha}_0^\mu \to -\tilde{\alpha}_0^\mu$. We can also assume $x^{25} \leftrightarrow \tilde{x}^{25}$. Under these two operations, we indeed have that $X_R \to -X_R$.

(f) The Neumann boundary conditions are already satisfied. There is no periodic boundary condition to be imposed, thus there is no winding modes for the open string.

(g) The interesting piece is

$$2\alpha' p^{25}\tau = \alpha' p^{25}(\sigma^+ + \sigma^-) \to \alpha' p^{25}(\sigma^+ - \sigma^-) = 2\alpha' p^{25}\sigma \tag{25}$$

The other parts of $\hat{X}^{25}$ are obtained similarily to the closed string

(h) Note that at $\sigma = 0, \pi$, $e^{-im\sigma^+} - e^{-im\sigma^-}$ vanishes. Thus, we have

$$\hat{X}^{25}(\pi) - \hat{X}^{25}(0) = 2\alpha' p^{25}\pi = 2\alpha'\frac{n}{R}\pi = 2\pi n\hat{R} \tag{26}$$

Since the right-hand-side is a multiple of 2π, it denotes the same space-time point. This mean that $X^{25}(\pi)$ and $X^{25}(0)$ are attached at the same space-time point $\hat{x}^{25}$, and the momentum number n denotes the number of times the open string winds around the compact direction.

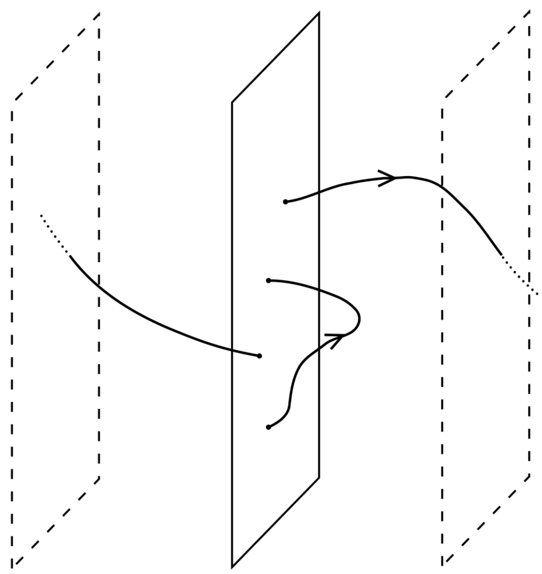

Figure 2: Open strings with Dirichlet boundary conditions. Here the dashed planes are identified. In our case, with only 1 compactified dimension, the plane is 25-dimensional, it is a D24-brane, and the transverse direction is 1-dimensional

(i) Taking the limit of the compactification $R \to 0$, one obtains that the T-dual radius $\hat{R} \to \infty$. Thus, if we consider a string theory with Neumann boundary conditions, i.e. string theory with a D25-brane, we compactify it in one dimension, and we take the radius to 0, this is equivalent to a string theory on uncompactified space living on D24-branes, i.e. with one Dirichlet boundary condition.

More generally, if we have a theory with Dp-branes, we can do this procedure of compactifying and taking $R \to 0$ on any direction. If we choose a direction transverse to the Dp-brane, i.e. a direction which had a Dirichlet boundary condition, we obtain back the Neumann condition and thus the brane becomes higher dimensional, we go to a D$(p+1)$-brane.[1] However, if we (compactify + $R \to 0$) a direction along the brane, we turn a Neumann b.c. to a Dirichlet b.c. and reduce to a $D(p-1)$-brane (as we did in the exercise).

[1]We usually say that the D$(p+1)$-branes *wraps* the corresponding compactification circle